MIDTERM COMPLEX ANALYSIS,
December 9th, 2019, 9:00am–11:00am,
Aletta Jacobshal 01, A1–I8.

Please provide complete arguments and/or calculations for each of your answers. The exam consists of 3 questions. You can score up to 6 points for each question, and you obtain 2 points for free.

In this way you will score in total between 2 and 20 points.

(1) For any $a \in \mathbb{C}$ let $f_a \colon \mathbb{C} \to \mathbb{C}$ be the function given by $f(z) = z^3 + 3z + a$.
   (a) (2 points.) Show that for $a \neq \pm 2i$ all zeros of $f(z)$ have multiplicity one.
   (b) (2 points.) In terms of $x, y \in \mathbb{R}$ such that $z = x + iy$, find the real part $u(x, y)$ of $f_a(z)$ and show by explicit calculation that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.
   (c) (2 points.) Now generalize the above as follows: assume $f \colon \mathbb{C} \to \mathbb{C}$ is analytic; you may assume that its real part $u(x, y)$ and its imaginary part $v(x, y)$ are sufficiently often differentiable with respect to both x and y. Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

(2) Consider $f \colon \mathbb{C} \to \mathbb{C}$ given by $f(z) = |z|^2$.
   (a) (2 points.) What is the range of the function f?
   (b) (2 points.) Prove that f is not differentiable in $z \neq 0$.
   (c) (2 points.) Prove that f is differentiable in $z = 0$.

(3) Let $D = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. We consider $\mathrm{Log} \colon D \to \mathbb{C}$ defined by $\mathrm{Log}(z) = \log(|z|) + i \arg(z)$.
   (a) (2 points.) Let γ be the semi-circle parametrized by $2e^{it}$, with $-\pi/2 \leq t \leq \pi/2$. Determine $\int_\gamma \mathrm{Log}(z)\, dz$.
   (b) (2 points.) For $\epsilon > 0$ (and < 2) let ℓ be the vertical line segment from $i\epsilon$ to $2i$. Determine $\int_\ell \mathrm{Log}(z)\, dz$.
   (c) (2 points.) Note that $(z\mathrm{Log}(z) - z)' = \mathrm{Log}(z)$. What is $\int_\Gamma \mathrm{Log}(z)\, dz$ if Γ is the closed loop consisting of γ, ℓ, the line segment from $-2i$ to $-i\epsilon$, and the semicircle in D from $-i\epsilon$ to $i\epsilon$?